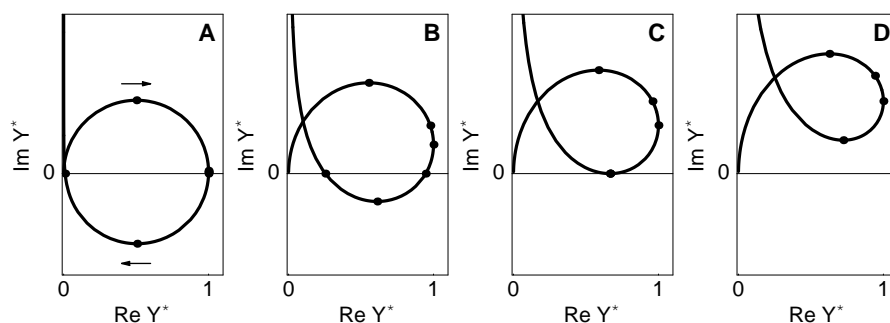


Handbook of Electrochemical Impedance Spectroscopy



CIRCUITS made of RESISTORS, INDUCTORS and CAPACITORS

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Chapter 1

Circuits made of R, L and C

1.1 (L+(R/C)) circuit

1.1.1 Circuit

Fig. 1.1.

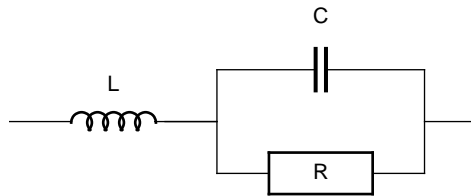


Figure 1.1: Circuit (L+(R/C)).

1.1.2 Impedance

$$Z(\omega) = Li\omega + \frac{R}{1 + RCi\omega}$$
$$\operatorname{Re} Z(\omega) = \frac{R}{C^2R^2\omega^2 + 1}, \operatorname{Im} Z(\omega) = \omega \left(L - \frac{CR^2}{C^2R^2\omega^2 + 1} \right)$$

1.1.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = iTu + \frac{1}{1 + iu}, \quad u = RC\omega, \quad T = \frac{L}{CR^2} \quad (1.1)$$

$$\operatorname{Re} Z^*(u) = \frac{1}{u^2 + 1}, \operatorname{Im} Z^*(u) = u \left(T - \frac{1}{u^2 + 1} \right)$$

Reduced characteristic angular frequency $u_c = 1$ with:

$$\operatorname{Re} Z(u_c) = 1/2, \operatorname{Im} Z(u_c) = T - 1/2$$

$T < 1 \Rightarrow$:

- $u_{\text{Im } Z=0} = \sqrt{\frac{1-T}{T}}$, $\text{Re } Z(u_{\text{Im } Z=0}) = T$.
- reduced angular frequency at the apex :

$$u_a = \frac{1}{\sqrt{2}} \sqrt{\frac{-2T + \sqrt{8T+1} - 1}{T}}$$

with:

$$\begin{aligned} \text{Re } Z(u_a) &= \frac{1}{4} (\sqrt{8T+1} + 1) \\ \text{Im } Z(u_a) &= \frac{\sqrt{T} (\sqrt{8T+1} - 3) \sqrt{-2T + \sqrt{8T+1} - 1}}{\sqrt{2} (\sqrt{8T+1} - 1)} \end{aligned}$$

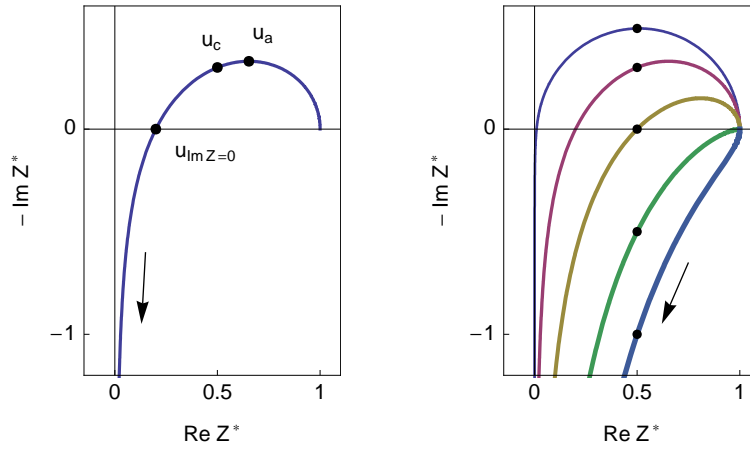


Figure 1.2: Nyquist diagram of the reduced impedance for the (L+(R/C)) circuit (Fig. 1.1, Eq. (1.2)) plotted for $T = 0.2$ (left) and $T = 0.01, 0.2, 0.5, 1, 1.5$ (right). The line thickness increases with increasing T . Dots: reduced characteristic angular frequency $u_c = 1$ (right).

1.2 ($R_0 + (L + (R/C))$) circuit

Fig. 1.3.

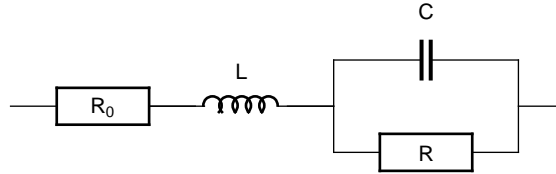


Figure 1.3: Circuit ($R_0 + (L + (R/C))$).

1.2.1 Impedance

$$Z(\omega) = R_0 + Li\omega + \frac{R}{1 + RCi\omega}$$

$$\operatorname{Re} Z(\omega) = R_0 + \frac{R}{C^2 R^2 \omega^2 + 1}, \operatorname{Im} Z(\omega) = \omega \left(L - \frac{CR^2}{C^2 R^2 \omega^2 + 1} \right)$$

1.2.2 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \rho + iTu + \frac{1}{1 + iu}, \quad u = RC\omega, \quad \rho = \frac{R_0}{R}, \quad T = \frac{L}{CR^2} \quad (1.2)$$

$$\operatorname{Re} Z^*(u) = \rho + \frac{1}{u^2 + 1}, \quad \operatorname{Im} Z^*(u) = u \left(T - \frac{1}{u^2 + 1} \right)$$

Reduced characteristic angular frequency $u_c = 1$ with:

$$\operatorname{Re} Z(u_c) = \rho + 1/2, \quad \operatorname{Im} Z(u_c) = T - 1/2$$

$T < 1 \Rightarrow$:

- $u_{\operatorname{Im} Z=0} = \sqrt{\frac{1-T}{T}}, \operatorname{Re} Z(u_{\operatorname{Im} Z=0}) = \rho + T.$
- reduced angular frequency at the apex :

$$u_a = \frac{1}{\sqrt{2}} \sqrt{\frac{-2T + \sqrt{8T+1} - 1}{T}}$$

with:

$$\operatorname{Re} Z(u_a) = \rho + \frac{1}{4} (\sqrt{8T+1} + 1)$$

$$\operatorname{Im} Z(u_a) = \frac{\sqrt{T} (\sqrt{8T+1} - 3) \sqrt{-2T + \sqrt{8T+1} - 1}}{\sqrt{2} (\sqrt{8T+1} - 1)}$$

1.3 (C+(R/L)) circuit

1.3.1 Circuit

Fig. 1.5.

1.3.2 Impedance

$$Z(\omega) = \frac{1}{Ci\omega} + \frac{LRi\omega}{R + Li\omega}$$

$$\operatorname{Re} Z(\omega) = \frac{L^2 R \omega^2}{L^2 \omega^2 + R^2}, \quad \operatorname{Im} Z(\omega) = \frac{L R^2 \omega}{L^2 \omega^2 + R^2} - \frac{1}{C\omega}$$

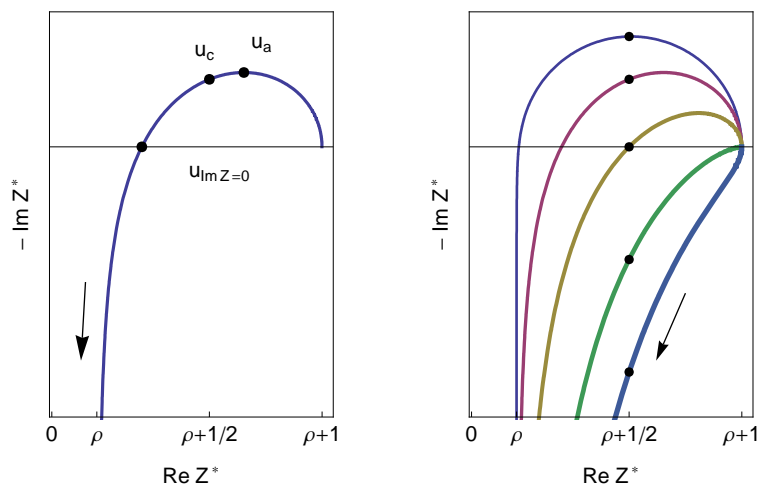


Figure 1.4: Nyquist diagram of the reduced impedance for the $(R_0+(L+(R/C)))$ circuit (Fig. 1.1, Eq. (1.2)) plotted $\rho = 0.2$ and $T = 0.2$ (left) and $T = 0.01, 0.2, 0.5, 1, 1.5$ (right). The line thickness increases with increasing T . Dots: reduced characteristic angular frequency $u_c = 1$ (right).

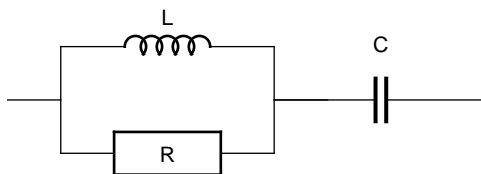


Figure 1.5: Circuit $(C+(R/L))$.

1.3.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \frac{1}{iT u} + \frac{i u}{1 + i u}, \quad u = \frac{L}{R} \omega, \quad T = \frac{R^2 C}{L} \quad (1.3)$$

$$\operatorname{Re} Z^*(u) = \frac{u^2}{u^2 + 1}, \quad \operatorname{Im} Z^*(u) = \frac{u}{u^2 + 1} - \frac{1}{T u}$$

Reduced characteristic angular frequency $u_c = 1$ with:

$$\operatorname{Re} Z(u_c) = 1/2, \quad \operatorname{Im} Z(u_c) = 1/2 - 1/T$$

$T > 1 \Rightarrow$:

- $u_{\operatorname{Im} Z=0} = \frac{1}{\sqrt{T-1}}, \quad \operatorname{Re} Z(u_{\operatorname{Im} Z=0}) = \frac{1}{T}$.
- reduced angular frequency at the apex :

$$u_a = \frac{1}{\sqrt{2}} \sqrt{\frac{T + \sqrt{T} + 8\sqrt{T} + 2}{T - 1}}$$

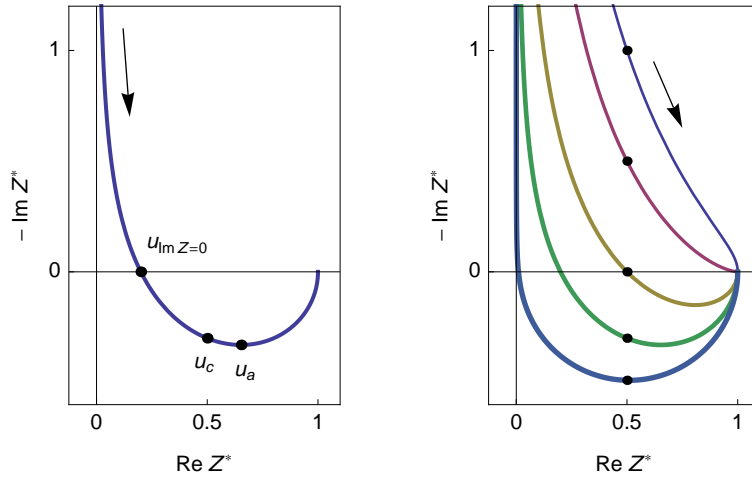


Figure 1.6: Nyquist diagram of the reduced impedance for the (C+(R/L)) circuit (Fig. 1.5, Eq. (1.3)) plotted for $T = 5$ (left) and $T = 0.66, 1, 2, 5, 100$ (right). The line thickness increases with increasing T . Dots: reduced characteristic angular frequency $u_c = 1$ (right).

with:

$$\operatorname{Re} Z(u_a) = \frac{1}{4} \left(\frac{1}{\sqrt{\frac{T}{T+8}}} + 1 \right)$$

$$\operatorname{Im} Z(u_a) = \frac{\sqrt{2}(T-1) \left(\sqrt{T} + \sqrt{T+8} \right)}{T \left(3\sqrt{T} + \sqrt{T+8} \right) \sqrt{\frac{T+\sqrt{T+8}\sqrt{T+2}}{T-1}}}$$

1.4 (R/L)+(R/C) circuit

Fig. 1.7.

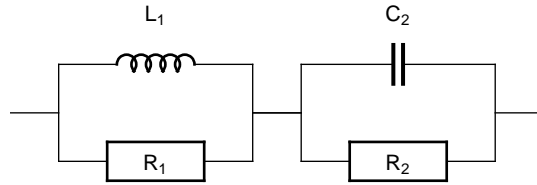


Figure 1.7: Circuit (R/L)+(R/C).

1.4.1 Impedance

$$Z(\omega) = \frac{L_1 R_1 i \omega}{L_1 i \omega + R_1} + \frac{R_2}{C_2 R_2 i \omega + 1} = \frac{R_1 \tau_1 i \omega}{1 + \tau_1 i \omega} + \frac{R_2}{1 + \tau_2 i \omega}, \quad \tau_1 = \frac{L_1}{R_1}, \quad \tau_2 = R_2 C_2 \quad (1.4)$$

$$\operatorname{Re} Z(\omega) = R_1 \left(1 - \frac{1}{\tau_1^2 \omega^2 + 1} \right) + \frac{R_2}{\tau_2^2 \omega^2 + 1}, \quad \operatorname{Im} Z(\omega) = \frac{R_1 \tau_1 \omega}{\tau_1^2 \omega^2 + 1} - \frac{R_2 \tau_2 \omega}{\tau_2^2 \omega^2 + 1}$$

$$\lim_{\omega \rightarrow 0} Z(\omega) = R_2, \quad \lim_{\omega \rightarrow \infty} Z(\omega) = R_1$$

1.4.2 Reduced impedance

$$Z^*(u) = \frac{Z(\omega)}{R_2} = \rho \frac{i u}{1 + i u} + \frac{1}{1 + T i u}, \quad u = \omega \tau_1, \quad \rho = \frac{R_1}{R_2}, \quad T = \frac{\tau_2}{\tau_1}$$

$$\operatorname{Re} Z^*(u) = \frac{u^2 \rho}{u^2 + 1} + \frac{1}{T^2 u^2 + 1}, \quad \operatorname{Im} Z^*(u) = \frac{u \rho}{u^2 + 1} - \frac{T u}{T^2 u^2 + 1}$$

1.4.3 Nyquist impedance diagrams

- $T > 1$, Fig. 1.8.

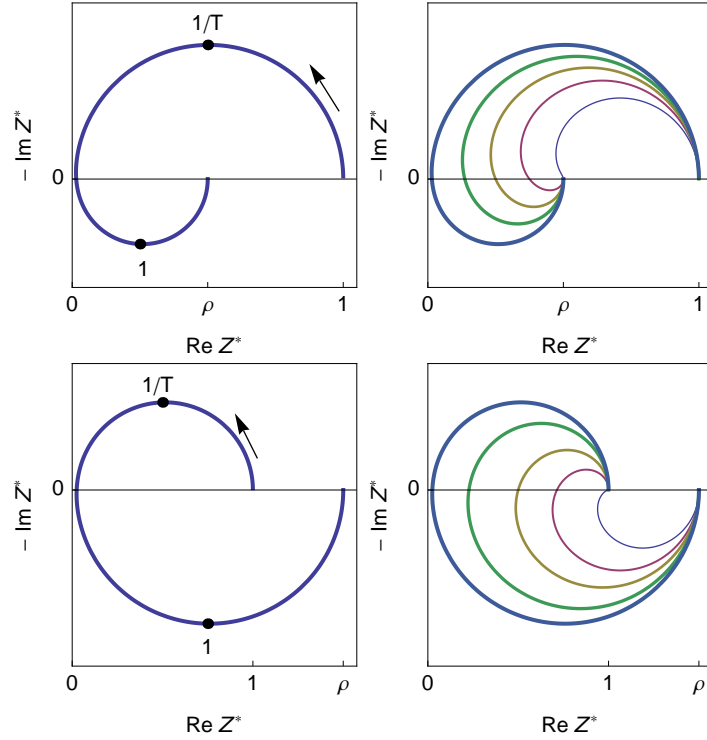


Figure 1.8: $T > 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit (Fig. 1.7, Eq. (1.4)) plotted for : top : $\rho < 1$ ($\rho = 0.5$), bottom : $\rho > 1$ ($\rho = 1.5$). $T \gg 1$ ($T = 10^2$) (left) and increasing values of T (right). The line thickness increases with increasing T .

- $T < 1$, Fig. 1.9.
- $T = 1$, Fig. 1.10.

$$T = 1 \Rightarrow Z^*(u) = \frac{1 + \rho i u}{1 + i u}$$

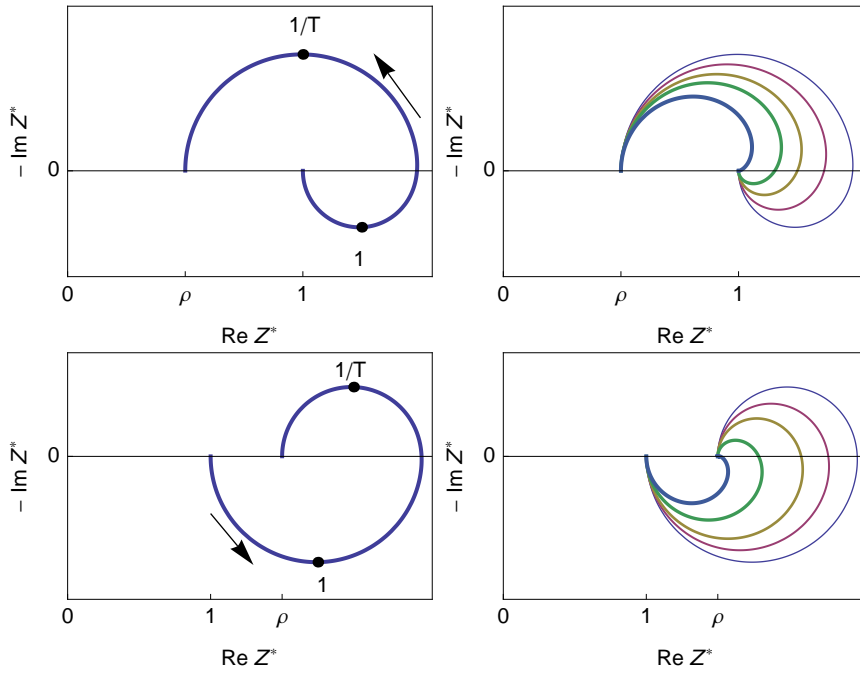


Figure 1.9: $T < 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit (Fig. 1.7, Eq. (1.4)) plotted for : top : $\rho < 1$ ($\rho = 0.5$), bottom : $\rho > 1$ ($\rho = 1.5$). $T \ll 1$ ($T = 10^{-2}$) (left) and increasing values of T (right). The line thickness increases with increasing T .

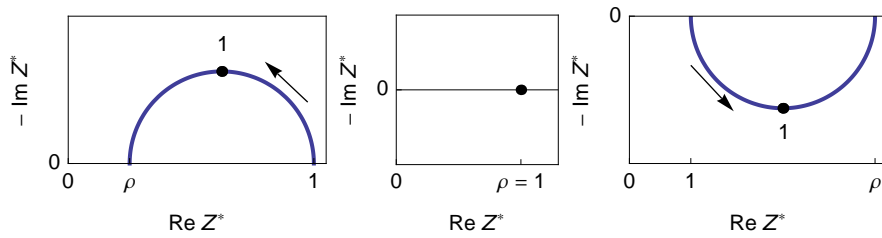


Figure 1.10: $T = 1$. Nyquist diagrams of the impedance for the (R/L)+(R/C) circuit. Left: $\rho < 1$, middle: $\rho = 1$, right: $\rho > 1$.

1.4.4 Inductive and capacitive Nyquist diagrams

$$T > 1 \text{ and } \frac{1}{T} < \rho < T \text{ or } T < 1 \text{ and } T < \rho < \frac{1}{T}$$

$$\Rightarrow u_{\text{Im}=0} = \frac{\sqrt{T - \rho}}{\sqrt{T(T\rho - 1)}}, \text{Re}_{\text{Im}=0} = \frac{1 + \rho}{1 + T}$$

Fig. 1.11.

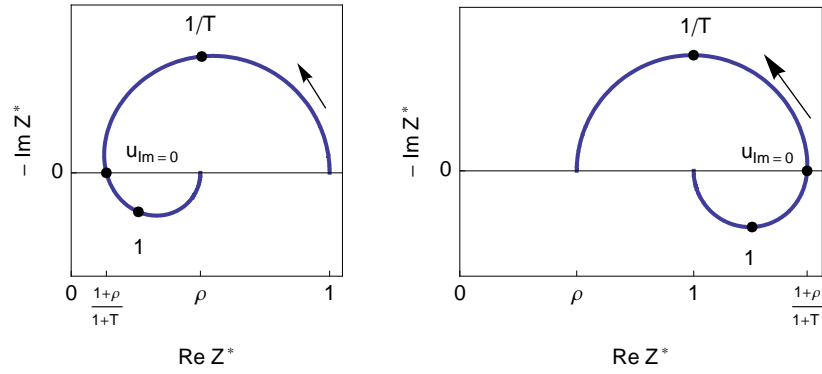


Figure 1.11: Inductive and capacitive Nyquist diagrams. Left : $T > 1$ and $\frac{1}{T} < \rho < T$, right : $T < 1$ and $T < \rho < \frac{1}{T}$.

1.4.5 $\rho = R_1/R_2 = 1$

$$Z^*(u) = \frac{i u}{1 + i u} + \frac{1}{1 + T i u}$$

Fig. 1.12.

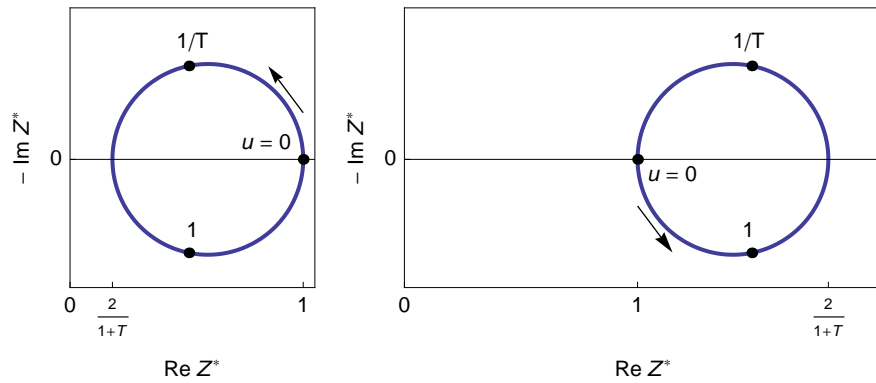


Figure 1.12: $\rho = R_1/R_2 = 1$. Nyquist diagram: full circle. Left: $T > 1$, right $T < 1$.

1.4.6 Array of Nyquist impedance diagrams

Fig. 1.13.

1.5 RLC parallel circuit

1.5.1 Circuit

Fig. 1.14.

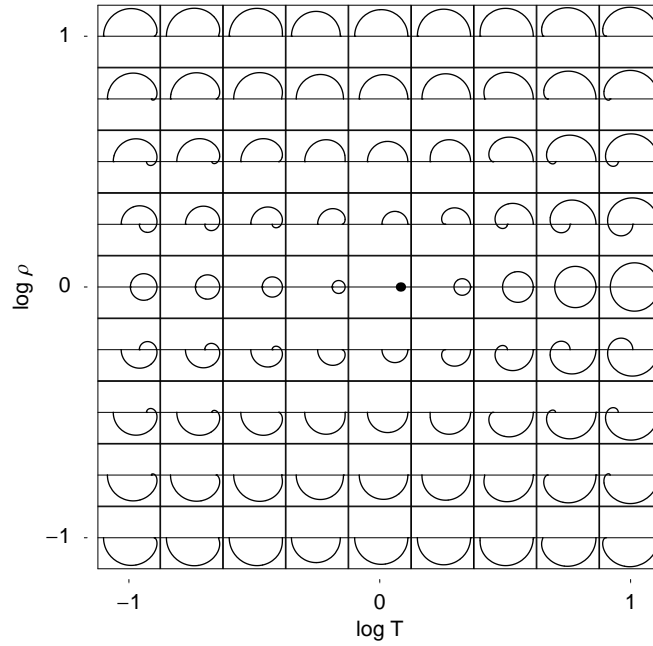


Figure 1.13: Array of Nyquist impedance diagrams for the (R/L)+(R/C) circuit. $T = \rho = 1 \Rightarrow Z^*(u) = 1, \forall u.$

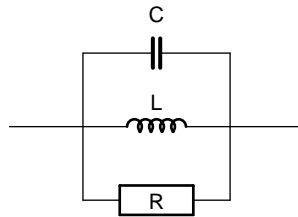


Figure 1.14: Circuit ((R/L)/C).

1.5.2 Admittance

$$Y(\omega) = \frac{1}{Li\omega} + Ci\omega + \frac{1}{R} = \frac{Li\omega + R + CLR(i\omega)^2}{LRi\omega}$$

$$\operatorname{Re} Y(\omega) = \frac{1}{R}, \operatorname{Im} Y(\omega) = -\frac{1}{L\omega} + C\omega$$

$\operatorname{Re} Y(\omega)$ is constant, $\lim_{\omega \rightarrow 0} \operatorname{Im} Y(\omega) = -\infty$, $\lim_{\omega \rightarrow \infty} \operatorname{Im} Y(\omega) = \infty \Rightarrow$ Nyquist diagram of $Y(\omega)$ is a vertical straight line.

1.5.3 Reduced admittance

$$Y^*(u) = RY(u) = 1 + \Lambda \left(iu + \frac{1}{iu} \right), \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}$$

$$\operatorname{Re} Y^*(u) = 1, \quad \operatorname{Im} Y^*(u) = \Lambda \left(u - \frac{1}{u} \right)$$

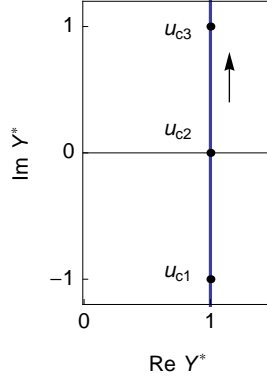


Figure 1.15: Nyquist diagram of the $((R/L)/C)$ circuit reduced admittance. $u_{c1} = (-1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$, $u_{c2} = 1$, $u_{c3} = (1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$.

1.5.4 Impedance

$$Z(\omega) = \frac{1}{Y(\omega)} = \frac{1}{\frac{1}{Li\omega} + Ci\omega + \frac{1}{R}} = \frac{LRi\omega}{Li\omega + R + CLR(i\omega)^2}$$

$$\operatorname{Re} Z(\omega) = \frac{L^2 R \omega^2}{L^2 \omega^2 + (R - CLR\omega^2)^2}, \quad \operatorname{Im} Z(\omega) = \frac{LR^2 \omega (1 - CL\omega^2)}{L^2 \omega^2 + R^2 (-1 + CL\omega^2)^2}$$

The Nyquist diagram of $Y(\omega)$ is a vertical straight line \Rightarrow the Nyquist diagram of $Z(\omega)$ is a full circle.

1.5.5 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = \frac{i u}{\Lambda + i u + \Lambda (i u)^2}, \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}$$

$$\operatorname{Re} Z^*(u) = \frac{u^2}{u^2 + \Lambda^2 (1 - u^2)^2}, \quad \operatorname{Im} Z^*(u) = \frac{\Lambda u (1 - u^2)}{u^2 + \Lambda^2 (1 - u^2)^2}$$

1.6 RLC serie circuit

1.6.1 Circuit

Fig. 1.17.

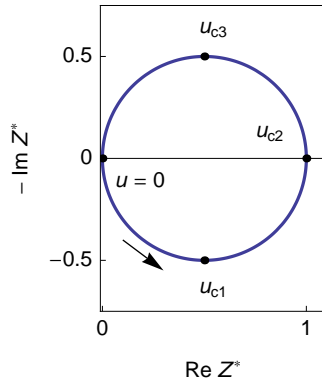


Figure 1.16: Nyquist diagram of the $((R/L)/C)$ circuit reduced impedance. $u_{c1} = (-1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$, $u_{c2} = u_r = 1$, $u_{c3} = (1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$ ($u_{c3} - u_{c1} = \Lambda$).

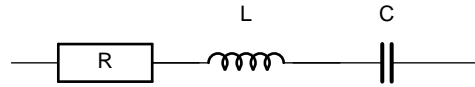


Figure 1.17: Circuit $((R+L)+C)$.

1.6.2 Impedance

$$Z(\omega) = R + Li\omega + \frac{1}{Ci\omega} = \frac{1 + RCi\omega + CL(i\omega)^2}{Ci\omega}$$

$$\operatorname{Re} Z(\omega) = R, \operatorname{Im} Z(\omega) = -\frac{1}{C\omega} + L\omega$$

$\operatorname{Re} Z(\omega)$ is constant, $\lim_{\omega \rightarrow 0} \operatorname{Im} Z(\omega) = -\infty$, $\lim_{\omega \rightarrow \infty} \operatorname{Im} Z(\omega) = \infty \Rightarrow$ Nyquist diagram of $Z(\omega)$ is a vertical straight line.

1.6.3 Reduced impedance

$$Z^*(u) = \frac{Z(u)}{R} = 1 + \frac{1}{\Lambda} \left(iu + \frac{1}{iu} \right), \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}$$

$$\operatorname{Re} Z^*(u) = 1, \quad \operatorname{Im} Z^*(u) = \frac{1}{\Lambda} \left(u - \frac{1}{u} \right)$$

1.6.4 Admittance

$$Y(\omega) = \frac{1}{Z(\omega)} = \frac{Ci\omega}{1 + RCi\omega + CL(i\omega)^2}$$

$$\operatorname{Re} Y(\omega) = -\frac{C^2 R \omega^2}{C^2 R^2 \omega^2 + (-1 + CL\omega^2)^2}, \quad \operatorname{Im} Y(\omega) = \frac{C\omega(1 - CL\omega^2)}{1 + C\omega^2(CR^2 + L(-2 + CL\omega^2))}$$

The Nyquist diagram of $Z(\omega)$ is a vertical straight line \Rightarrow the Nyquist diagram of $Y(\omega)$ is a full circle.

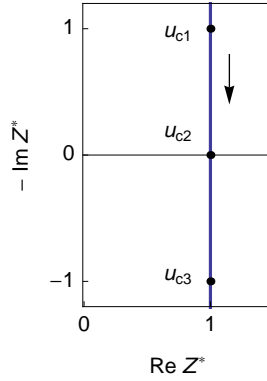


Figure 1.18: Nyquist diagram of the ((R+L)+C) circuit reduced impedance. $u_{c1} = (-\Lambda + \sqrt{4 + \Lambda^2})/2$, $u_{c2} = 1$, $u_{c3} = (\Lambda + \sqrt{4 + \Lambda^2})/2$.

1.6.5 Reduced admittance

$$Y^*(u) = RY(u) = \frac{\Lambda i u}{1 + \Lambda i u + (i u)^2}, \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}$$

$$\operatorname{Re} Y^*(u) = \frac{u^2 \Lambda^2}{1 + u^4 + u^2 (-2 + \Lambda^2)}, \quad \operatorname{Im} Y^*(u) = \frac{u \Lambda (1 - u^2)}{1 + u^4 + u^2 (-2 + \Lambda^2)}$$

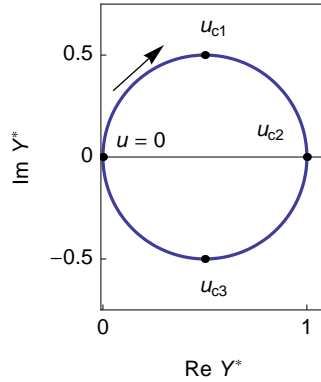


Figure 1.19: Nyquist diagram of the ((R+L)+C) circuit reduced admittance. $u_{c1} = (-\Lambda + \sqrt{4 + \Lambda^2})/2$, $u_{c2} = u_r = 1$, $u_{c3} = (\Lambda + \sqrt{4 + \Lambda^2})/2$, $(u_{c3} - u_{c1} = \Lambda)$.

1.7 $R_0 + \text{RLC}$ parallel circuit

1.7.1 Circuit

Fig. 1.20.

1.7.2 Impedance

$$Z(\omega) = R_0 + \frac{L R i \omega}{L i \omega + R + C L R (i \omega)^2} \quad (1.5)$$

1.8. TRANSFORMATION FORMULAE $(R_1/L_1)+(R_1/C_2) \rightarrow R_1 + \text{RLC PARALLEL}$ 17

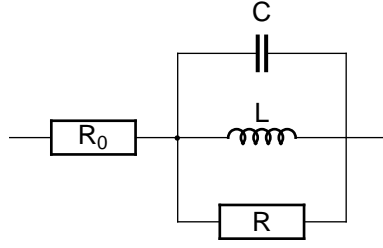


Figure 1.20: $R_0 + \text{RLC}$ parallel circuit.

$$Z^*(u) = \frac{Z(u)}{R} = \rho + \frac{i u}{\Lambda + i u + \Lambda (i u)^2}, \quad \rho = \frac{R_0}{R}, \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}$$

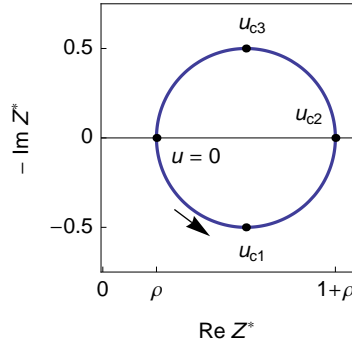


Figure 1.21: Nyquist reduced impedance diagram of the $R_0 + \text{RLC}$ parallel circuit. $u_{c1} = (-1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$, $u_{c2} = u_r = 1$, $u_{c3} = (1 + \sqrt{1 + 4\Lambda^2})/2\Lambda$ ($u_{c3} - u_{c1} = \Lambda$).

1.8 Transformation formulae $(R_1/L_1)+(R_1/C_2) \rightarrow r_1 + \text{RLC parallel}$

$r_1 + r_2/l_2/c_2$ parallel circuit is not-distinguishable from $(R_1/L_1)+(R_1/C_2)$ circuit for $R_1^2 C_2/L_2 > 1$ (Fig. 1.22).

$$Z(p) = \frac{R_1 \left(C_2 L_1 p^2 + \frac{2L_1 p}{R_1} + 1 \right)}{C_2 L_1 p^2 + \frac{p(C_2 R_1^2 + L_1)}{R_1} + 1} \quad (1.6)$$

$$z(p) = \frac{r_1 \left(c_2 l_2 p^2 + \frac{l_2 p (r_1 + r_2)}{r_1 r_2} + 1 \right)}{c_2 l_2 p^2 + \frac{l_2 p}{r_2} + 1} \quad (1.7)$$

$$c_2 = \frac{C_2 L_1}{L_1 - C_2 R_1^2}, \quad l_2 = L_1 - C_2 R_1^2, \quad r_2 = R_1 \left(\frac{2L_1}{C_2 R_1^2 + L_1} - 1 \right) \quad (1.8)$$

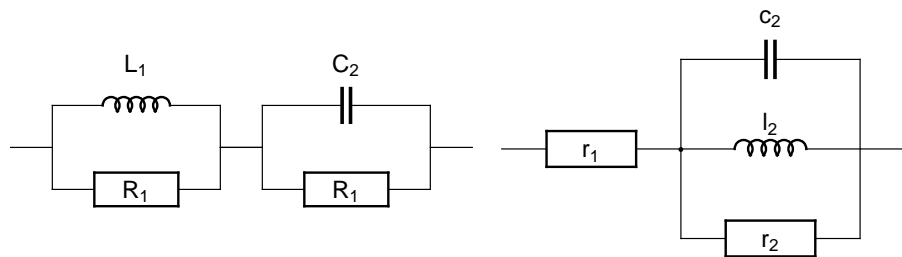


Figure 1.22: $(R_1/L_1) + (R_1/C_2)$ ($(R_1/L_1) + (R_2/C_2)$) circuit with $R_1 = R_2$ and $r_1 + r_2/l_2/c_2$ parallel circuit.

Chapter 2

Quartz resonator

2.1 BVD equivalent circuit

Fig. 2.1, [3, 5, 6, 1].

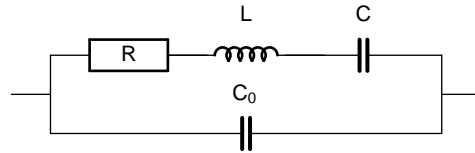


Figure 2.1: BVD (Butterworth-van Dyke)-equivalent circuit of a quartz resonator.

2.2 Admittance

$$Y(\omega) = \frac{1}{R + Li\omega + \frac{1}{Ci\omega}} + i\omega C_0 = i\omega \left(\frac{C}{1 + Ci\omega(Li\omega + R)} + C_0 \right)$$

$$\operatorname{Re} Y(\omega) = \frac{C^2 R \omega^2}{C^2 R^2 \omega^2 + (-1 + CL\omega^2)^2}, \quad \operatorname{Im} Y(\omega) = \omega \left(\frac{C(1 - CL\omega^2)}{1 + C\omega^2(CL R^2 + L(-2 + CL\omega^2))} + C_0 \right)$$

2.3 Reduced admittance

$$Y^*(u) = RY(u) = \frac{\Lambda i u}{1 + \Lambda i u + (i u)^2} + \gamma i u, \quad u = \omega \sqrt{LC}, \quad \Lambda = R \sqrt{\frac{C}{L}}, \quad \gamma = \frac{RC_0}{\sqrt{LC}}$$

$$\operatorname{Re} Y^*(u) = \frac{u^2 \Lambda^2}{1 + u^4 + u^2(-2 + \Lambda^2)}, \quad \operatorname{Im} Y^*(u) = u \gamma + \frac{u \Lambda (1 - u^2)}{1 + u^4 + u^2(-2 + \Lambda^2)}$$

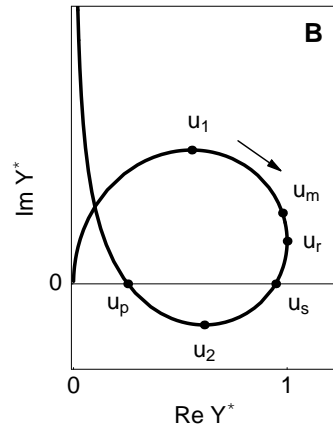


Figure 2.2: Definitions for u_1 , u_m , u_r , u_s , u_2 and u_p .

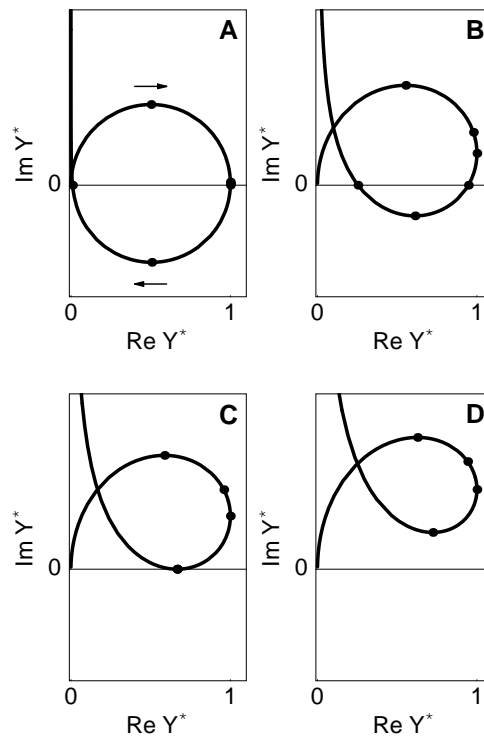


Figure 2.3: Change of admittance diagram with γ . $\Lambda = 1$, $\gamma = 10^{-2}$ (A), 2×10^{-1} (B), $1/3$ (C), $1/2$ (D).

2.3.1 Characteristic frequencies

- Maximum of the real part of Y^* for:
 $u_r = 1 \Rightarrow \text{Re } Y^*(u_r) = 1, \text{Im } Y^*(u_r) = \gamma$

- Zero-phase reduced angular frequencies: u_s and u_p defined for $\gamma < 1/(2 + \Lambda)$:

$$1. \quad \gamma < \frac{1}{2 + \Lambda} \Rightarrow$$

$$u_s = \sqrt{\frac{\Lambda - \gamma(-2 + \Lambda^2) - \Lambda \sqrt{1 - 2\gamma\Lambda + \gamma^2(-4 + \Lambda^2)}}{2\gamma}},$$

$$\operatorname{Re} Y^*(u_s) = \frac{1}{2} \left(1 + \gamma\Lambda + \sqrt{(\gamma(\Lambda - 2) - 1)(\gamma(\Lambda + 2) - 1)} \right)$$

$$u_p = \sqrt{\frac{2\gamma + \Lambda - \gamma\Lambda^2 + \Lambda \sqrt{1 - 2\gamma\Lambda + \gamma^2(-4 + \Lambda^2)}}{2\gamma}},$$

$$\operatorname{Re} Y^*(u_p) = \frac{1}{2} \left(1 + \gamma\Lambda - \sqrt{(\gamma(\Lambda - 2) - 1)(\gamma(\Lambda + 2) - 1)} \right)$$

$$2. \quad \gamma = \frac{1}{2 + \Lambda} \Rightarrow$$

$$u_s = u_p = \frac{-\gamma\Lambda^2 + 2\gamma + \Lambda}{2\gamma}$$

$$\operatorname{Re} Y^*(u_s) = \operatorname{Re} Y^*(u_p) = \frac{1}{2}(1 + \gamma\Lambda) \quad (\text{Fig. 2.3C}).$$

$$3. \quad \gamma > \frac{1}{2 + \Lambda} \Rightarrow$$

no zero-phase reduced angular frequency (Fig. 2.3D).

Real quartz : $C_0 \approx 10^{-12}$ F, $C \approx 10^{-14}$ F, $R \approx 100$ Ω , $L \approx \times 10^{-2}$ H $\Rightarrow \Lambda \approx 10^{-4}$ and $\gamma \approx 10^{-2}$ [4, 2].

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